



A novel lifetime analysis of repairable systems via Daubechies wavelets

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Abstract

It is well-known that minimally repaired systems are characterized by non-homogeneous Poisson processes. When the underlying lifetime distribution function or its associated intensity function is completely known, the statistical inference of the cumulative number of failures/repairs is made through some standard techniques. On the other hand, when the parametric form of the lifetime distribution or the intensity function is not known, non-parametric inference techniques such as non-parametric maximum likelihood estimate (NPMLE) and kernel-based estimates are employed. Since these techniques are based on the underlying lifetime data, the resulting estimates of the cumulative number of failures/repairs with respect to the system operation time are discontinuous and troublesome in assessing the reliability measures such as reliability function and mean time to failure, etc. In this paper, we propose a novel lifetime analysis of the repairable systems via Daubechies wavelets. More specifically, we generalize the seminal Daubechies wavelet estimate by Kuhl and Bhairgond (Winter Simul Conf Proc 1:562–571, 2000) by combining the ideas on NPMLE and KMBs. In both simulation experiments and real data analyses, we investigate applicability of our Daubechies wavelet-based approaches and compare them with the conventional non-parametric methods.

Keywords Repairable systems · Non-homogeneous poisson processes · Non-parametric inference · Daubechies wavelets · Simulation experiments · Real data analyses

1 Introduction

Lifetime analysis is quite beneficial to understand the failure mechanism of engineering systems by means of the theoretical statistics and can be applied to design their maintenance planning (Barlow & Hunter, 1960; Barlow & Proschan, 1975). In general, there are two

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types of systems in the lifetime analysis; repairable systems and non-repairable systems. Especially, the repairable system reliability modeling plays a central role in reliability engineering practices (Ascher & Feingold, 1984), because non-repairable systems are maintained by the so-called perfect repair, which is equivalent to replace the failed units/components by new ones. That is to say, the perfect repair offers a new unit which is as good as new when the operating unit fails. On the other hand, the minimal repair only restores the failed unit to its functioning condition just prior to the failure, where the minimal repair means the age (in the sense of effectiveness) of the unit will not be changed after a repair activity. Since the seminal contribution by Barlow and Hunter (1960), a number of researchers have discussed the minimal repair models and their mathematical features. Nakagawa and Kowada (1983) used the failure rate to define a minimally repaired process, and gave an insight to deal with the minimal repair process mathematically. Aven (1983) considered the optimal replacement policies under a minimal repair assumption. Brown and Proschan (1983) introduced the imperfect repair by combining the minimal repair with the perfect repair. Block et al. (1985) extended the notion of imperfect repair by introducing the time dependency. Kijima (1989) introduced the general repair and succeeded to represent a wider class of repair activities. Although the general repair is a well-defined stochastic model to describe the repair activities, its analytical treatment is rather troublesome. In other words, due to the analytical difficulty of the general repair model, it has not been frequently used in the industrial modeling. Instead, since the minimal repair approximately represents a variety of repair effects under a plausible assumption, i.e., it only restores the failed unit to the latest functioning condition just prior to the failure, the research for the minimal repair has been still conducted in some literatures (Aven & Jensen, 2000; Finkelstein, 2004; Tadj et al., 2011).

In this paper we concern the statistical inference problem of the minimal repair process, which denotes the cumulative number of failures/minimal repairs. It is well known that the minimal repair process is given by a non-homogeneous Poisson process (NHPP). Hence, the statistical inference problem of the minimal repair process is equivalent to one for the NHPPs. There are two statistical estimation approaches for the NHPPs; parametric estimation and non-parametric estimation. For the representative parametric models in the NHPPs, the power law (Weibull process) model by Duane (1964) and Cox and Lewis model (Cox & Lewis, 1966) have been often used in the lifetime analysis. Lee and Lee (1978) and Crow (1982) gave the fundamental results for the power law model; point estimation based on the maximum likelihood method and the confidence intervals, respectively. Muralidharan et al. (2006, 2007) derived the predictive distribution of the power law model and the conditional interval estimate for the ratio of intensity parameters, respectively. Hence, when the intensity function is not known in advance, the parametric estimation does not work in the real data analyses. On the other hand, it is worth mentioning that the intensity function to describe the field lifetime data cannot be known in many cases. In such cases, any non-parametric estimation will be applied to the lifetime analysis. Boswell (1966) and Zielinski et al. (1993) considered the constrained non-parametric maximum likelihood estimate (NPMLE) as an extension of the naive (natural) estimate of NHPPs. Diggle and Marron (1988) noticed the equivalence of estimations in the failure rate function and the intensity function, and developed the kernel-based estimation (KBE) method. Saito et al. (2014); Saito and Dohi (2016) applied both parametric and non-parametric bootstrapping techniques to estimate the optimal preventive replacement times with minimal repair, respectively. Apart from the lifetime data analysis of hardware products/equipment, the NHPPs have been used traditionally to described the software-fault count processes in software system testing (Lyu, 1996). Since the seminal contribution by Goel and Okumoto (1979), a number of parametric software reliability growth models based on NHPPs have been proposed in the literature (Abdel-Ghaly et al., 1986; Achcar et al.,

1998; Goel, 1985; Gokhale & Trivedi, 1998; Ohba, 1984; Ohishi et al., 2009; Okamura et al., 2013; Yamada et al., 1983; Zhao & Xie, 1996). Wang et al. (2007) and Dohi et al. (2020) developed several non-parametric methods based on the KBE. Saito and Dohi (2015) applied the constrained NPMLE to the software fault-count data.

In this paper, we concern the lifetime analysis of repairable systems via the Daubechies wavelets (Daubechies, 1992). Recently, the wavelet-based approach has received much attentions for statistical inference of stochastic processes. Especially, the discrete Haar wavelet (Haar, 1909) has been used as alternative non-parametric estimation for NHPPs. Xiao and Dohi (2013) combined the existing data-transformation techniques and denoising algorithms, and established the Haar wavelet shrinkage estimation for NHPP-based software reliability models, where the underlying failure data were the fault-detection time-interval (group) data. In the lifetime analysis, however, it is common to deal with the continuous time domain data which are based on the failure-occurrence time data. For such continuous data, Kuhl and Bhairgond (2000) proposed a Daubechies wavelet estimate for the intensity function of NHPP. Xiao and Dohi (2012) applied the Kuhl and Bhairgond estimate to analyze software failure-detection time data. The above wavelet-based methods could give the better goodness-of-fit performances than the common parametric estimation methods. On one hand, in the paper by Kuhl and Bhairgond (2000), their Daubechies wavelet estimate was not be well defined.

In this paper, we show that the Kuhl and Bhairgond estimation can be interpreted as a finite improvement over the well-known naive intensity function estimation. In this paper, we demonstrate that the Kuhl and Bhairgond estimate (Kuhl & Bhairgond, 2000) can be interpreted as a finite improvement over the well-known naive intensity function estimation. Moreover, we propose an estimator that is more generalized compared to the Kuhl and Bhairgond estimation, with the latter being a special case of the estimator we propose. In addition, we introduce two Daubechies wavelet estimators, which can be considered as an extension and enhancement of the NPMLE and KBEs. So, we provide a generalization framework of the Daubechies wavelets estimate of NHPPs by Kuhl and Bhairgond (2000). In simulation experiments, we compare our Daubechies wavelet approaches with the classical parametric and non-parametric estimation methods, and investigate the goodness-of-fit throughout the real data analyses with both hardware failure/repair time data and software fault count data.

2 Repairable systems

Minimal repair and replacement are frequently used as practical maintenance activities for real engineering systems, where the minimal repair is a maintenance activity to repair the failed component, so that its function is recovered without changing its age, while a replacement restores the entire component into the new condition, so that it behaves as a new component. Many researchers (Abdel-Ghaly et al., 1986; Achcar et al., 1998; Goel, 1985; Gokhale & Trivedi, 1998; Ohba, 1984; Ohishi et al., 2009; Okamura et al., 2013; Yamada et al., 1983; Yamada & Osaki, 1983; Zhao & Xie, 1996) considered the repair of many software or hardware failures as a minimal repair. Suppose at the moment that there are two components, where the first (second) component has the absolutely continuous lifetime distribution $F(t)$ ($G(t)$) with $F(0) = 0$ and $F(\infty) = 1$ ($G(0) = 0$ and $G(\infty) = 1$). When the first component fails and the failed component is replaced by the second one, the distribution function of the

time to failure of the second of two components is given by the Stieltjes convolution;

$$F * G(t) = \int_0^t F(t-u) dG(u), \quad (1)$$

where the replacement component is assumed to be new on the installation. On the other hand, when one replaces the failed component by one with equal age, the survivor function of the first and second components is given by the relevation transform (see Krakowski, 1973);

$$F \circ G(t) = \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(u)} dF(u), \quad (2)$$

where in general $\bar{\psi}(\cdot) = \psi(\cdot)$. Barlow and Hunter (1960) and Barlow and Proschan (1975) summarized the notion of overhaul with minimal repair for any intervening failures, where the failure rate of component remains undisturbed by any repair of failures.

If the replacement is made with identical components, say $F(t) = G(t)$ in Eq.(1), and the time to replace a failed component by new one is negligible, then the cumulative number of failures/replacements by time t is given by the renewal counting process $N(t) = \max\{n : \sum_{i=1}^n T_i \leq t\}$, where $F(t) = P\{T_i \leq t\}$ ($i = 1, 2, \dots$). On the other hand, if the minimal repair is made at each failure and the time to repair an failed component is negligible, then the cumulative number of failures/repairs by time t is, from $F(t) = G(t)$ in Eq.(2), given by the non-homogeneous Poisson process (NHPP) (Baxter, 1982);

$$P(N(t) = n) = \frac{\Lambda(t; \theta)^n}{n!} e^{-\Lambda(t; \theta)}, \quad (3)$$

where

$$\Lambda(t; \theta) = E[N(t)] = -\log \bar{F}(t; \theta) = \int_0^t \lambda(u; \theta) du \quad (4)$$

is the mean value function with the parameter vector θ , $F(t) = F(t; \theta)$, and $\lambda(t; \theta) = d\Lambda(t; \theta)/dt$ is called the intensity function.

The commonly used technique to estimate the parameter θ in $\Lambda(t; \theta)$ and $\lambda(t; \theta)$ is the maximum likelihood estimation. Suppose that n failure/repair times $\mathbf{t} = (t_1, t_2, \dots, t_n)$ with right truncation at T ($\geq t_n$) are observed for respective components, where the time to complete each repair is negligible. Then, the log likelihood function is given by

$$LLF(\theta; \mathbf{t}) = \sum_{i=1}^n \lambda(t_i; \theta) - \Lambda(T; \theta). \quad (5)$$

By maximizing $LLF(\theta; \mathbf{t})$ with respect to θ , we get the maximum likelihood estimate $\hat{\theta}$.

In the parametric lifetime analysis, it is common to assume the power law model (Duane, 1964) and Cox and Lewis model (Cox & Lewis, 1966). In the power-law model, the intensity function is given by $\lambda(t; \theta) = abt^{b-1}$, $\theta \in (a, b)$. It is known as a flexible model with increasing intensity ($b > 1$) and decreasing intensity ($0 < b < 1$) function. When $b = 1$, it is reduced to a homogeneous Poisson process (HPP) with constant intensity. In Cox and Lewis model (Cox & Lewis, 1966), the intensity function is given by $\lambda(t; \theta) = -\exp(a + bt)$, $\theta \in (a, b)$. Because the maximum likelihood method provides an unbiased estimate in many cases, if we have an enough number of data and can know the real intensity form, it is possible to infer the failure/repair process with minimal repair accurately. Of course, since knowing the real intensity function in advance is difficult in general, the parametric method above has some limitations.

3 Non-parametric estimation of NHPP

Consider the case where the intensity function $\lambda(t)$ is completely unknown. In this case, the method of parametric maximum likelihood in Sect. 2 does not work under the incomplete knowledge on the intensity function and/or mean value function. The most intuitive but simplest method to estimate the intensity function is a piecewise-linear interpolation, which is called the naive (natural) estimate. For the n failure time data, t_i ($i = 1, 2, \dots, n$) with the right-truncation T ($\geq t_n$), define

$$\hat{\lambda}(t) = \begin{cases} \frac{1}{t_i - t_{i-1}}, & t_0 \leq t \leq t_i; \quad i = 1, \dots, n, t_0 = 0, \\ \frac{1}{T - t_n}, & t_n \leq t \leq T. \end{cases} \quad (6)$$

Then we have the following step-function estimate with breakpoints t_i :

$$\hat{\Lambda}(t) = \int_0^t \hat{\lambda}(x) dx = \begin{cases} i + \frac{t - t_i}{t_{i+1} - t_i}, & t_i \leq t \leq t_{i+1}; \quad i = 0, 1, \dots, n-1, \\ n + \frac{0.5(t - t_n)}{T - t_n}, & t_n \leq t \leq T. \end{cases} \quad (7)$$

The resulting naive estimate of the mean value function $\hat{\Lambda}(t)$ in Eq.(7) is obtained by plotting n failure points and connecting them by line segments. Since only one sample path, t_i ($i = 0, 1, 2, \dots, n$), is available in a single failure time series data, it seems to be the straightforward but the most natural estimate of the cumulative number of failures, because the mean squares error between the native estimate and the underlying lifetime data is always zero. In the case where the failure time sequences of multiple items are independent and identically distributed, the native estimate is constructed by a superposition of the multiple lifetime data sets. However, it should be noted that the above estimate of the intensity function is discontinuous everywhere and tends to fluctuate with big noise.

Boswell (1966) and Zielinski et al. (1993) considered the constrained non-parametric maximum likelihood estimate (NPMLE). Let T_{ji} ($j = 1, 2, \dots, l; i = 1, 2, \dots, n$) denote the random variable of i -th failure-occurrence time data at j -th NHPP with realizations t_{ji} , and let $T^* = \max_{1 \leq j \leq l} T_{jn}$ with realization t^* . Then, the normalized random variable can be represented by $X_{ji} = T_{ji}/T^* \in (0, 1]$ with realizations $x_{ji} = t_{ji}/t^* \in (0, 1]$. Arranging the normalized failure-occurrence time data into a single list $x_0 (= 0) < \dots < x_f (= 1)$ with the total number of data f , the constrained NPMLE with multiple failure data sets are obtained by

$$\hat{\lambda}(x) = \begin{cases} 0, & 0 \leq x < x_1, \\ \hat{\lambda}_i, & x_i \leq x < x_{i+1}; \quad i = 1, 2, \dots, f-1, \end{cases} \quad (8)$$

where

$$\hat{\lambda}_i = \max_{1 \leq s \leq i} \min_{i \leq k \leq f} \frac{\sum_{j=s}^k m_j}{\sum_{j=s}^k \Delta_j}, \quad (9)$$

$$\Delta_i = \sum_{j=1}^l \max \left\{ 0, \min \{ (x_n - x_i), (x_{i+1} - x_i) \} \right\} \quad (10)$$

and m_i means the number of failures occurred at time x_i . Frankly speaking the constrained NPMLE is defined as the upper bound of the naive estimate. It often overestimates the underlying NHPP (Saito & Dohi, 2016).

Next, we introduce the KBE methods. Define

$$\hat{\lambda}(x | \chi) = \frac{1}{h} \sum_{i=1}^n K\left(\frac{x - x_i}{h}\right), \quad (11)$$

where $K(\cdot)$ denotes a kernel function and $h (> 0)$ is a positive constant, called the smoothing parameter or bandwidth. It is well known that the choice of h is more sensitive rather than the choice of kernel function for the accuracy of $\lambda(x)$. In this paper we assume the following five kernel functions;

$$K_1(x) = (1 - |x|)I_{[-1,1]}(x), \quad (\text{Triangle}) \quad (12)$$

$$K_2(x) = \frac{3}{4}(1 - x^2)^2 I_{[-1,1]}(x), \quad (\text{Epanechnikov}) \quad (13)$$

$$K_3(x) = \frac{15}{16}(1 - x^2)^2 I_{[-1,1]}(x), \quad (\text{Quartic}) \quad (14)$$

$$K_4(x) = \frac{70}{81}(1 - \|x\|^3)^3 I_{[-1,1]}(x), \quad (\text{tricube}) \quad (15)$$

$$K_5(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad (\text{Gaussian}) \quad (16)$$

where

$$I_{[-1,1]}(x) = \begin{cases} 1 & \text{for } x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

The kernel-based approach approximates the intensity function with a superposition of kernel functions with location parameter at each failure time. The main reason why absolutely continuous non-parametric estimate is preferred is that the naive estimate in Eq.(6) and the constrained NPMLE in Eq.(8) are functions of $\chi = (x_1, x_2, \dots, x_n)$. The promising approach to estimate the bandwidth h is the least-squares cross-validation (LSCV) by Diggle and Marron (1988), which minimizes the relevant integrated least squares error. Suppose that the failure/repair time data are divided into *training data* and *validation data*. In the situation where n failure/repair time data based on unknown NHPP are given, it is possible to make n training data sets with $n - 1$ failure data, by leaving out one of each i -th ($= 1, 2, \dots, n$) data from the original data x_i ($i = 1, 2, \dots, n$) and each of the leaved i -th ($= 1, 2, \dots, n$) data is regarded as the validation data. The integrated least squares error of the intensity function $\hat{\lambda}(x)$ in LSCV method is defined by

$$\text{ISE}(h) = \int_0^1 \{\hat{\lambda}(x; \chi) - \lambda(x; \chi)\}^2 dx. \quad (18)$$

After removing independent terms of h in Eq.(18), it can be checked that the optimal bandwidth minimizing $\text{ISE}(h)$ is equal to h minimizing the following function;

$$\text{CV}(h) = \int_0^1 \hat{\lambda}(x; \chi)^2 dx - 2 \sum_{j=1}^n \hat{\lambda}_{h,j}(x_i; \chi), \quad (19)$$

where, from Eq.(11),

$$\hat{\lambda}_{h,j}(x; \chi) = \frac{1}{h} \sum_{i=1, i \neq j}^n K\left(\frac{x - x_j}{h}\right). \quad (20)$$

Note that the above non-parametric estimates are not smooth everywhere and are still troublesome in the analysis. In the following section we summarize the Daubechies wavelet approach with the aim at smoothing the non-parametric estimates, and generalize the existing Daubechies wavelet estimate of NHPPs by Kuhl and Bhairgond (2000).

4 Wavelet-based approach

4.1 Daubechies wavelets

Daubechies (1992) proposed a set of continuous and compactly supported wavelets, which are very popular in wavelet analysis field. The Daubechies wavelets are not defined in closed form, where the Daubechies scaling function and wavelet function are defined in the following forms;

$$\phi(t) = \sum_{i=0}^l h_i \phi(2t - i), \quad (21)$$

$$\psi(t) = \sum_{i=0}^l (-1)^i h_{n-i} \psi(2t - i). \quad (22)$$

In Eqs.(21) and (22), the filter coefficients h_i are given in the reference (Daubechies, 1992), and l is the support width and determines the smoothness of the functions $\phi(t)$ and $\psi(t)$. The starting values $\{\phi(t), t = 1, 2, \dots, n-1\}$ can be obtained by solving the recursive formula;

$$\begin{cases} \sum_{i=0}^n \phi(t) = 1, \\ \phi(0) = 0, \\ \phi(n) = 0, \\ \phi(t) = \sum_{i=0}^n h_i \phi(2t - i), \quad t = 1, 2, \dots, n-1. \end{cases}$$

The other values of $\phi(t)$ with $t \in [0, n]$ and $t \neq 1, 2, \dots, n-1$ are calculated in Eq. (21).

4.2 Wavelet estimate

Dissimilar to discrete Haar wavelets (Haar, 1909), Daubechies wavelets (Daubechies, 1992) have the smooth and continuous basis functions except at $n = 1$ in Eqs.(21) and (22). So, the Daubechies wavelets can approximate the intensity function of an arbitrary NHPP, $\lambda(t)$, which is assumed to be an absolutely continuous function. Since the Daubechies scaling function $\phi(t)$ in Eq.(21) can take negative values, one needs a positive basis function for approximating a positive $\lambda(t)$. Walter and Shen (1998) developed a positive basis function for estimating probability density functions.

Let $\phi(t)$ be the Daubechies scaling function having the compact support. The positive basis function by Walter and Shen (1998) is given by

$$P_r(t) = \sum_{j \in \mathbb{Z}} r^{\|j\|} \phi(t - j) \quad (23)$$

with parameter r satisfying $a \leq r < 1$, where $\mathbb{Z}$ is a set of all integers. The parameter r controls the minimum value of $P_r(t)$, so that the minimum value of $P_r(t)$ is greater than or equal to 0 when $r = a$ (> 0). Figure 1 illustrates the positive basis functions with $r = 0.1$

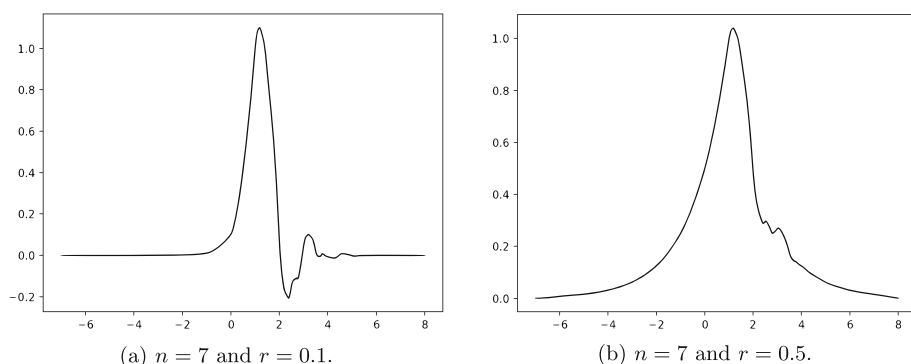


Fig. 1 Positive basis functions

and $r = 0.5$ for $\phi(t)$ with $n = 7$. In this case, we can see that the minimum value of $P_r(t)$ is less than 0 when $r = 0.1$.

By using the positive basis function, $P_r(t)$, a positive reproducing kernel, $k_r(t, s) \in V_0$, is given by

$$k_r(t, s) = \left(\frac{1-r}{1+r} \right)^2 \sum_{a=-\infty}^{\infty} P_r(t-a) P_r(s-a). \quad (24)$$

Let $k(t, s)$ denote a reproducing kernel satisfying

$$\int_{-\infty}^{\infty} k(t, s) \lambda(s) ds = \lambda(t), \quad (25)$$

where $\lambda(t)$ is an arbitrary continuous intensity function. For $\lambda(t) \in L_2(\mathbb{R})$, an approximation of the function $\lambda_0(t) \in V_0$ is constructed as

$$\lambda_0(t) \approx \int_{-\infty}^{\infty} k_r(t, s) \lambda(s) ds. \quad (26)$$

In general, the approximation of an arbitrary function $\lambda(t) \in V_m$ is given by

$$\lambda_m(t) = \int_{-\infty}^{\infty} k_{r,m}(t, s) \lambda(s) ds, \quad (27)$$

where $k_{r,m}(t, s)$ in Eq.(27) is the positive reproducing kernel in V_m and is given by

$$k_{r,m}(t, s) = 2^m k_r(2^m t, 2^m s). \quad (28)$$

Kuhl and Bhairgond (2000) proposed a wavelet estimate based on Eq.(6) and Eq. (27) when $T = t_n$ in Eq.(6) as follows:

$$\hat{\lambda}_{r,m}(t; \mathbf{t}) = 2^m \left(\frac{1-r}{1+r} \right)^2 \sum_{a=-k}^k \left\{ \sum_{i=1}^n P_r(2^m t_i - a) \right\} P_r(2^m t - a), \quad (29)$$

where t_i ($i = 1, 2, \dots, n$) are the failure/repair times, the parameter a is determined in the range in which the positive basis function covers the entire failure/repair times, and the resolution level m is determined based on the detail of the approximation. Unfortunately, Kuhl and Bhairgond (2000) did not clarify the derivation procedure of their wavelet estimate

in Eq.(29). Here, we derive the same result to complete the discussion, and improve Kuhl and Bhairgond's estimate (Kuhl & Bhairgond, 2000).

Let $\lambda_{naive}(t; \mathbf{t})$ denote the well-known naive estimate of NHPP intensity function;

$$\hat{\lambda}_{naive}(t; \mathbf{t}) = \sum_{i=1}^n \frac{1}{t_i - t_{i-1}} I_i(t), \quad (30)$$

where

$$I_i(t) = \begin{cases} 1, & t_{i-1} < t \leq t_i \\ 0, & \text{otherwise.} \end{cases} \quad (31)$$

Substituting the naive estimate $\lambda(t; \mathbf{t})$ in Eq. (30) into Eq.(27), we can obtain the naive wavelet estimate (NWE);

$$\begin{aligned} \hat{\lambda}_{NWE}(t; \mathbf{t}) &= \int_{-\infty}^{\infty} k_{r,m}(t, s) \hat{\lambda}_{naive}(s) ds \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} k_{r,m}(t, s) \frac{1}{t_i - t_{i-1}} ds \\ &= 2^m \left(\frac{1-r}{1+r} \right)^2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sum_{a=-\infty}^{\infty} Pr(2^m t - a) \\ &\quad \times Pr(2^m s - a) \frac{1}{t_i - t_{i-1}} ds \\ &= 2^m \left(\frac{1-r}{1+r} \right)^2 \sum_{a=-\infty}^{\infty} \left\{ \sum_{i=1}^n \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \right. \\ &\quad \left. \times Pr(2^m s - a) ds \right\} Pr(2^m t - a). \end{aligned} \quad (32)$$

Hence it is seen that the exact form of NWE contains the integral parts. If each integral is approximated by an elementary rectangular approximation method, it is given by

$$\begin{aligned} \hat{\lambda}_{RNWE}(t; \mathbf{t}) &= 2^m \left(\frac{1-r}{1+r} \right)^2 \sum_{a=-\infty}^{\infty} \left\{ \sum_{i=1}^n \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \right. \\ &\quad \left. \times Pr(2^m t_i - a) ds \right\} Pr(2^m t - a) \\ &= 2^m \left(\frac{1-r}{1+r} \right)^2 \sum_{a=-\infty}^{\infty} \left\{ \sum_{i=1}^n \frac{1}{t_i - t_{i-1}} (t_i - t_{i-1}) \right. \\ &\quad \left. \times Pr(2^m t_i - a) \right\} Pr(2^m t - a) \\ &= 2^m \left(\frac{1-r}{1+r} \right)^2 \sum_{a=-\infty}^{\infty} \left\{ \sum_{i=1}^N Pr(2^m t_i - a) \right\} \\ &\quad \times Pr(2^m t - a). \end{aligned} \quad (33)$$

We call the above estimate the rectangular approximate naive wavelet estimate (RNWE), which is equivalent to Kuhl and Bhairgond's estimate (Kuhl & Bhairgond, 2000). Strictly speaking, RNWE is an approximation of NWE from the computational point of view. In order to compute the NWE, $\hat{\lambda}_{NWE}(t)$, more accurately, we need to apply any numerical integration algorithm for Eq.(32).

As we have already shown in the above, both NWE and RNWE are based on the naive estimate in Eq.(30) of NHPP. However, it is also well known that the naive estimate is the most intuitive estimator but causes the overfitting problem. In other words, the native estimate lacks the generalization ability as a statistical estimator. In this paper, as alternative baseline estimates of NHPP, we employ the constrained NPMLE and the KBE. When $T = t_n$, Boswell (1966) proposed the following NPMLE of the intensity function by maximizing the upper bound of the empirical log likelihood function;

$$\hat{\lambda}_{NPMLE}(t; \mathbf{t}) = \begin{cases} 0, & 0 \leq t < t_1, \\ r_k(t), & t_k \leq t < t_{k+1}, \\ r_{n-1}(t), & t = t_n, \\ \infty, & t_n < t, \end{cases} \quad (34)$$

where

$$r_k(t) = \min_{v \geq k+1} \max_{u \leq k} \left[\frac{v - u}{J(u, v)} \right],$$

$$J(u, v) = \sum_{i=u+1}^v \{(n - i + 1)(t_i - t_{i-1})\}.$$

Then we define the non-parametric maximum likelihood wavelet estimate (NPMLWE) by

$$\hat{\lambda}_{NPMLWE}(t; \mathbf{t}) = \int_{-\infty}^{\infty} k_{r,m}(t, s) \hat{\lambda}_{npml}(s; \mathbf{t}) ds. \quad (35)$$

On the other hand, the KBE of NHPP intensity function is given by

$$\hat{\lambda}_{kernel}(t; \mathbf{t}) = \frac{1}{h} \sum_{i=1}^n K\left(\frac{t - t_i}{h}\right). \quad (36)$$

Then we define the kernel-based wavelet estimate (KBWE) by

$$\hat{\lambda}_{KBWE}(t; \mathbf{t}) = \int_{-\infty}^{\infty} k_{r,m}(t, s) \hat{\lambda}_{kernel}(s; \mathbf{t}) ds. \quad (37)$$

The basic idea here is also to denoise the kernel functions by the Daubechies wavelet.

In the above ways, we develop the corresponding Daubechies wavelet-based approaches to the well-known non-parametric estimation methods in the literature. In the following sections, we take place the simulation experiments and real data analyses to investigate our wavelet-based inference schemes.

5 Simulation experiments

5.1 Experimental setup

In this section, we carry out the Monte Carlo simulations and estimate the mean value function characterizing the minimal repair process with the common parametric/non-parametric methods and our wavelet-based methods. Our concern here is to compare the Daubechies wavelet approaches with the existing non-parametric methods and the parametric maximum

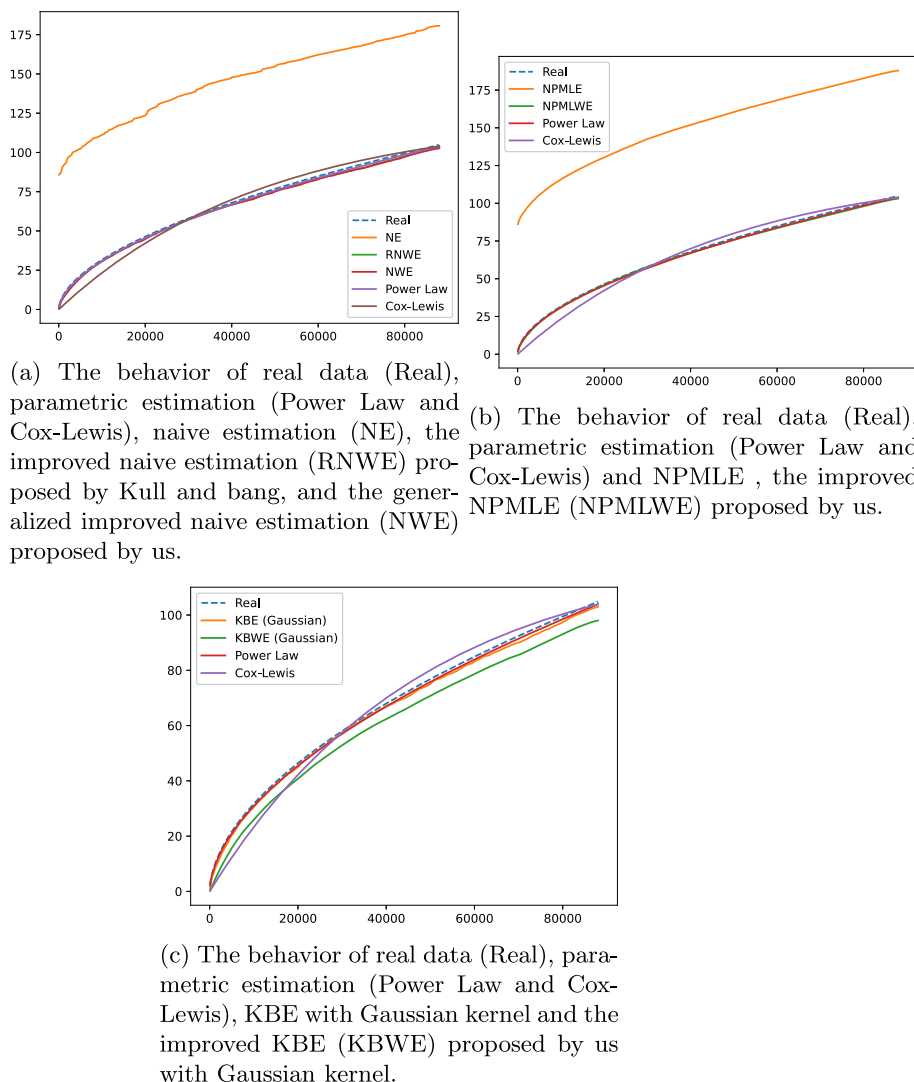


Fig. 2 Estimation of the cumulative number of failures/repairs in simulation experiments

likelihood methods for a representative minimal repair processes in terms of the goodness-of-fit. For the parametric models, we assume the power law model (Duane, 1964) and Cox and Lewis model (Cox & Lewis, 1966) as the baseline model, and apply the maximum likelihood method to estimate the model parameters. For non-parametric methods, we apply the naive estimate (NE), NPMLE, KBEs, and their associated Daubechies wavelet methods; NWE, RNWE, NPMLWE, KBWE, to estimate the intensity function of the power law NHPP. For both the KBEs and KBWEs, we assume five kernel functions in Eqs.(12)–(16).

In the Monte Carlo simulations, it is assumed that the mean value function of a baseline NHPP is given by the power law model with $\lambda_{real}(t; \theta) = at^b$ for the time interval $(0, T]$, with $a = 0.20$, $b = 0.55$ and $T = 88,000$. We use the standard linear congruential generator (<https://www.itl.nist.gov/div898/handbook/>) to generate the pseudo random numbers of

Table 1 Comparison of estimation methods based on MAE in simulation experiments

Model	MAE
<i>Parametric approach</i>	
Power law	0.93
Cow-lewis	3.31
<i>Existing non-parametric approach</i>	
NE	78.22
NPMLE	83.75
KBE (Gaussian)	1.49
KBE (Epanechnikov)	1.49
KBE (tricube)	1.48
KBE (Triangle)	1.50
KBE (Quartic)	1.47
Best RNWE	1.50
<i>Our wavelet approach</i>	
Best NWE	1.72
Best NPMLWE	0.96
Best KBWE (Gaussian)	5.93
Best KBWE (Epanechnikov)	6.76
Best KBWE (tricube)	6.44
Best KBWE (Triangle)	6.20
Best KBWE (Quartic)	6.41
Bold value indicates the best result in Parametric approach, Existing non-parametric approach or Our wavelet approach	

failure/repair times. Finally we generate 30 sets of failure/repair time sequence, and estimate the number of minimal repairs when 30 independent and identical components are running in parallel.

As a goodness-of-fit performance measure, we apply the mean absolute error (MAE);

$$\text{MAE} = \frac{\sum_{i=1}^k \|\Lambda(\tau_i) - \tilde{\Lambda}(\tau_i)\|}{k}, \quad (38)$$

where τ_i is the i -th equidistant sample point, $k = 1000$ is the number of the sample points, $\tilde{\Lambda}(\tau_i)$ is an estimate of the mean value function at $t = \tau_i$. For the parametric models (Cox & Lewis, 1966; Duane, 1964), once we obtain the model parameters, we can directly calculate the estimated value of $\Lambda(t)$ at an arbitrary time t . However, for the non-parametric methods, especially the wavelet-based methods, we obtain only an estimate of the intensity function at each τ_i . Therefore, we need to calculate estimated of the mean value function from the intensity estimates. Specifically, for the non-parametric methods, we take k equidistant sample points in the range of $(0, T]$ to obtain k sample values of the intensity function, and approximate an estimate of the mean value function at each τ_i by means of numerical integration based on the trapezoidal area between adjacent sample points. Based on the assumption that 30 independent and identical components run in parallel, we observe 30 failure/repair time sequences with the same right-truncation point T , and superpose 30 failure/repair time sequences into a single time sequence. So, we use the single time series data for the estimation.

In the Daubechies wavelet-based approaches, the parameter j in the positive basis function $P_r(t)$ can take all values in $\mathbb{Z}$. Xiao and Dohi (2012) gave a useful range for j empirically

as $j \in [-7, 7]$ in their experiments on software failure-detection time analysis. Since the function $P_r(t)$ decays to zero quickly, we adjust it as $j \in [-7, 8]$ even in our experiments. The range of the parameter s in $\hat{\lambda}_{r,m}(t)$ is determined so as to satisfy that the positive basis function covers the entire failure/repair times. Xiao and Dohi (2012) also took the range of s as

$$s \in [-(\text{Integer part of } 2^m t_n + 7), \text{ Integer part of } 2^m t_n + 7], \quad (39)$$

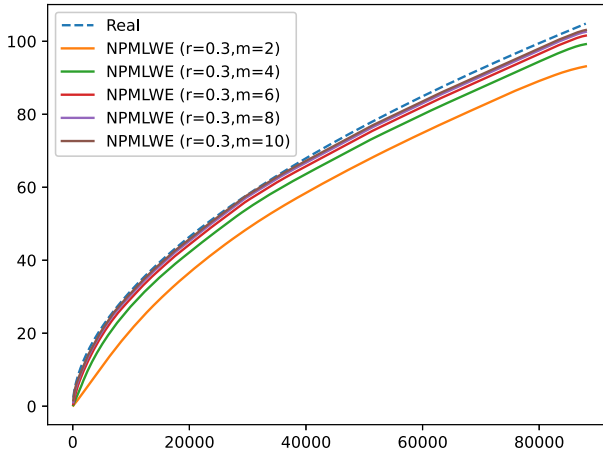
where t_n is the n -th failure/repair time.

5.2 Discussions

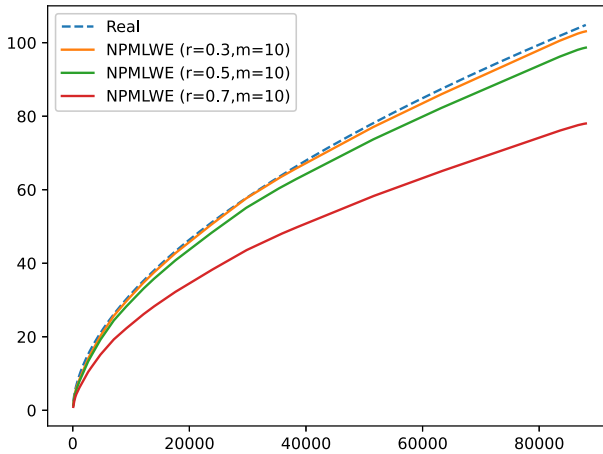
In the simulation experiments, we compare the representative parametric models (Cox & Lewis, 1966; Duane, 1964), the existing non-parametric methods with four wavelet shrinkage estimates; NWE, RNWE, NPMLWE and KBWE in terms of goodness-of-fit, where we turn up the parameters as $r = 0.3, 0.5, 0.7$ and $m = 2, 3, 4, 5, 6, 7, 8, 9, 10$, and the support width of Daubechies wavelet is given by 7. Figure 2 shows the dependence of the turning parameters r and m in the cumulative number of failures/repairs on the operation time, where each set of methods; (a) NE, RNWE, NWE and RNWE, (b) NPMLE and NPMLWE, and (c) KBE and KBWE, are compared against the parametric models (Power Law, Cox-Lewis) and the real intensity function.

For KBWEs, we select the best kernel function with the smallest MAE. In Fig. 2a, we can see that all RNWE, NWE and Power Law could fit the true mean value function (Real) very well. It is seen that the goodness-of-fit result by Cox-Lewis model is obviously poor, so the parametric model selection is essential to get accurate goodness-of-fit result. Also it is observed that the naive estimate (NE) deviates from the true mean value function significantly and poorly works for the estimation. This is due to the highly discontinuous nature of the intensity function by NE, so that the estimation error may be caused by the relatively rapid frequency of fluctuated intensity estimates by NE with respect to the sampling frequency of equidistant sample points. After applying the wavelet method to NE, we could smooth the NE by RNWE and NWE, and achieve the high estimation accuracy under the equidistant sample point spacing. In Fig. 2b, we observe the similar trend for NPMLE. Although the estimate by NPMLE was relatively smoother than that by NE, NPMLE is still discontinuous everywhere. Hence, even when NPMLWE could fit the true mean value function better, the estimation result by NPMLE deviated significantly from the true mean value function. In addition, Fig. 2c shows that the kernel-based estimate, especially, based on Gaussian kernel function, could fit the true mean value function. However, in the kernel-based approaches, the wavelet method, KBWE, could not give accurate result than KBE. This is a quite interesting observation. In fact, the use of kernel functions motivates us to smooth the discontinuous estimates such as the naive estimate. Wavelet methods can be interpreted as a smoothing technique. They can improve the roughness of estimators. However, since kernel estimation itself is a smooth estimator, applying wavelet methods again does not further improve its smoothness, and therefore, it actually leads to a reduction in estimation performance. That is to say, the denoising the kernel-based estimates by the Daubechies wavelet does not always lead to the better estimation results.

Table 1 compares the mean absolute error MAE by all estimation methods in our experiments. We examined the best combination of the resolution levels m and different tuning parameters r in the wavelet-based approach. First of all, it can be seen that Power Law gave the minimum MAE = 0.93. This is a natural result. Because when the form of the real



(a) The behavior of NPMLWE when the parameter r is fixed for different m parameters.



(b) The behavior of NPMLWE when the parameter m is fixed for different r parameters.

Fig. 3 Dependence of parameters r and m in the cumulative number of failure/repairs in simulation experiments

failure/repair model is known in advance, the maximum likelihood estimation is theoretically the optimal estimation result. But when the model selection fails as Cox-Lewis model, the goodness-of-fit performance becomes poor. The classical non-parametric methods; NE and NPMLE, gave worst estimation results, but the kernel-based methods with five kinds of kernel functions and the original Daubechies wavelet estimate by Kuhl and Bhairgond (2000) provided stable estimation results around $MAE = 1.50$. Next, we focus on our wavelet-based approach proposed in this paper. It can be confirmed that NPMLWE was the best non-parametric method with $MAE = 0.96$ which is very closed to the result $MAE = 0.93$ when the underlying intensity function is completely known. Also we could checked that our Daubechies wavelet estimates can improve the basic NE and NPMLE effectively. In Table 1,

the result of NWE was obtained based on a trapezoidal rule in numerical integrations. In other words, the approximation in RNWE employed by Kuhl and Bhairgond (2000) did work better than our straightforward approach NWE with numerical integrations.

Next, we concern the sensitivity of parameters m and r in the wavelet estimation approaches. Figure 3 shows the dependence of the turning parameters in the estimation results of cumulative number of failures/minimal repairs in NPMLWE, where different resolution levels m and tuning parameters r are assumed. In Fig. 3a and b, we change the value of m and r , respectively, and can know that these parameters strongly depend on the estimation results by NPMLWE. More precisely, we calculate the MAE values for all the wavelet-based approaches with varying values of m and r in Table 2. We can observe that all the wavelet estimation methods exhibited the similar increasing trend in terms of the tuning parameter r , but the value of MAE_1 gradually decreased as r decreased. On the resolution level m , it is seen that there is an optimal value such that the MAE is minimized. In our experiments, we empirically found that optimal resolution level are located in the range of $m \geq 8$.

6 Real data analyses

6.1 Data setup

In this section, we perform the real data analyses to investigate the goodness-of-fit performances of our non-parametric wavelet methods. Table 3 presents the data sets used in the analyses, where DS1 ~ DS4 (Lyu, 1996) are the software fault-count data sets observed in the system testing phases, and DS5 (Meeker & Escobar, 1998) is the lifetime data of vehicle. For the real data analyses, we assume eleven representative parametric models in Table 4 in addition to two parametric models (Power Law, Cox-Lewis), which are used for software reliability growth modeling. The goodness-of-fit is compared in terms of the mean absolute error. In the real data analysis, as one cannot know the non-parametric estimates except n data points, the mean absolute error in the real data analyses is modified as

$$MAE = \frac{\sum_{i=1}^n \|i - \tilde{\Lambda}(t_i)\|}{n}, \quad (40)$$

where n is the number of failures, t_i is the i -th failure-occurrence time, and $\tilde{\Lambda}(t_i)$ is an estimate of the mean value function at time $t = t_i$.

6.2 Discussions

Figure 4 illustrates the behavior of the estimation results by different estimation methods in DS1. Similar to the simulation experiments, we find that each subfigure compares the parametric models with minimum MAE and the best non-parametric method with minimum MAE with our wavelet-based methods with best tuning parameters r and resolution level m . In Fig. 4a, we can see that the behavior of the naive estimate is also similar to that in the simulation experiments, and that the estimated mean value function significantly deviates from the actual dataset, due to the discontinuity of the intensity estimate. On one hand, RNWE and NWE could significantly improve the underlying naive estimate. In DS1, it is seen that the best parametric method could give the better goodness-of-fit performance. In Fig. 4b, although NPMLE is smoother than naive estimate, it is still discontinuous, and deviates from the actual data, while the wavelet method NPMLWE improved the NPMLE. In Fig. 4c, the

Table 2 Sensitivity of model parameters m and r for the wavelet estimation approaches in simulation experiments

$m \backslash r$	0.3	0.5	0.7	$m \backslash r$	0.3	0.5	0.7
<i>RNWE</i>				<i>NWE</i>			
2	10.02	14.96	26.3	2	11.03	17.79	30.44
3	6.77	10.3	20.9	3	7.92	12.59	25.4
4	4.7	7.29	17.44	4	5.31	9.3	22.65
5	3.39	5.38	15.2	5	3.87	7.46	20.56
6	2.53	4.17	13.8	6	2.85	6.05	19.35
7	1.91	3.36	12.91	7	2.13	5.25	18.53
8	1.5	2.78	12.28	8	1.72	4.68	17.99
9	1.73	2.49	11.86	9	1.86	4.41	17.66
10	2.48	2.59	11.67	10	2.32	4.43	17.49
<i>NPMLWE</i>				<i>KBWE (Gaussian)</i>			
2	9.94	15.91	29.56	2	10.91	16.46	29.88
3	6.7	11.48	24.95	3	8.24	12.44	25.49
4	4.63	8.64	22.03	4	6.87	10.19	22.95
5	3.32	6.85	20.19	5	6.27	9.11	21.63
6	2.46	5.73	19.04	6	6.04	8.67	21.06
7	1.86	4.96	18.31	7	5.96	8.52	20.86
8	1.47	4.43	17.79	8	5.93	8.47	20.8
9	1.21	4.09	17.45	9	5.93	8.46	20.79
10	0.96	3.9	17.27	10	5.93	8.46	20.81
<i>KBWE (Epanechnikov)</i>				<i>KBWE (Tricube)</i>			
2	11.21	16.63	29.95	2	11.02	16.51	29.89
3	8.7	12.74	25.65	3	8.45	12.54	25.53
4	7.48	10.66	23.23	4	7.19	10.41	23.06
5	6.99	9.72	22.05	5	6.67	9.43	21.84
6	6.82	9.38	21.59	6	6.5	9.08	21.35
7	6.77	9.28	21.44	7	6.46	8.97	21.2
8	6.76	9.25	21.41	8	6.44	8.95	21.17
9	6.76	9.25	21.42	9	6.44	8.95	21.18
10	6.76	9.26	21.44	10	6.44	8.96	21.2
<i>KBWE (Triangle)</i>				<i>KBWE (Quartic)</i>			
2	11.06	16.57	29.93	2	11.08	16.56	29.92
3	8.45	12.6	25.59	3	8.5	12.61	25.58
4	7.11	10.4	23.09	4	7.21	10.45	23.11
5	6.53	9.34	21.8	5	6.67	9.45	21.86
6	6.31	8.93	21.26	6	6.48	9.06	21.35
7	6.24	8.78	21.07	7	6.43	8.95	21.19
8	6.21	8.73	21.01	8	6.41	8.92	21.15
9	6.2	8.72	21	9	6.41	8.92	21.15
10	6.2	8.72	21.02	10	6.41	8.92	21.18

Bold value indicates the best result in RNWE/NWE/NPMLWE/KBWE(Gaussian)/KBWE(Epanechnikov)/KBWE(Tricube)/KBWE(Triangle)/KBWE(Quartic)

Table 3 Data sets

Data set	Observation length	No. failures	Source	Type of failures
DS1	88682.000	136	SYS1 Lyu (1996)	Software failure
DS2	102594.000	104	SYS2 Lyu (1996)	Software failure
DS3	108708.000	86	S2 Lyu (1996)	Software failure
DS4	15369.500	54	CSR3 Lyu (1996)	Software failure
DS5	25.518	71	Meeker and Escobar (1998)	Hardware failure

Table 4 Representative parametric models in NHPP

Mean value function	$\Lambda(t)$ ($\omega, b, c > 0$)
Exp Goel and Okumoto (1979)	$\Lambda(t) = \omega(1 - e^{-bt})$
Gamma Yamada et al. (1983), Zhao and Xie (1996)	$\Lambda(t) = \omega \int_0^t \frac{c^b s^{b-1} e^{-cs}}{\Gamma(b)} ds$
Pareto Abdel-Ghaly et al. (1986)	$\Lambda(t) = \omega(1 - (\frac{c}{t+c})^b)$
TruncNormal Okamura et al. (2013)	$\Lambda(t) = \omega \frac{F(t) - F(0)}{1 - F(0)},$ $F(t) = \frac{1}{\sqrt{2\pi}b} \int_t^\infty e^{-\frac{(s-c)^2}{2b^2}} ds$
LogNormal Achcar et al. (1998); Okamura et al. (2013)	$\Lambda(t) = \omega \frac{1}{\sqrt{2\pi}b} \int_{\log(t)}^\infty e^{-\frac{(s-c)^2}{2b^2}} ds$
TruncLogist Ohba (1984)	$\Lambda(t) = \omega \frac{F(t) - F(0)}{1 - F(0)},$ $F(t) = \frac{1}{1 + e^{-\frac{t-c}{b}}}$
LogLogist Gokhale and Trivedi (1998)	$\Lambda(t) = \omega \frac{1}{1 + e^{-\frac{\log(t)-c}{b}}}$
TruncEVMMax Ohishi et al. (2009)	$\Lambda(t) = \omega \frac{F(t) - F(0)}{1 - F(0)},$ $F(t) = \omega(e^{-e^{-\frac{t-c}{b}}})$
LogEVMMax Ohishi et al. (2009)	$\Lambda(t) = e^{-e^{-\frac{c - \log(t)}{b}}}$
TruncEVMMin Ohishi et al. (2009)	$\Lambda(t) = \omega \frac{F(0) - F(-t)}{F(0)},$ $F(t) = e^{-e^{-\frac{t-c}{b}}}$
LogEVMMin Goel (1985)	$\Lambda(t) = \omega(1 - e^{-e^{\frac{\log(t)+c}{b}}})$

KBWE was compared with the KBE. Table 5 presents the goodness-of-fit performance of all the methods on different datasets, where “NULL” implies that the estimates of model parameters did not converge and the corresponding mean absolute errors were extremely large. We can observe that the NWE could achieve the better goodness-of-fit performance than the RNWE by Kuhl in four out of five datasets (DS1, DS2, DS4, DS5), and also had the best goodness-of-fit performance among all the estimation methods. This result is quite different from the simulation results in Sect. 5. In other words, it can be concluded that our wavelet-based method, NWE, can provide the best goodness-of-fit performances by turning

Fig. 4 Estimation of the cumulative number of failures/repairs in DS1

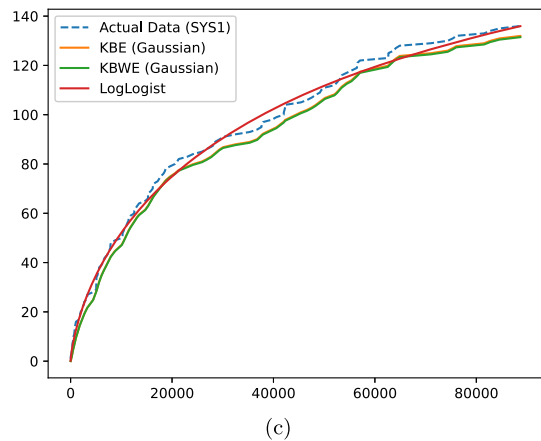
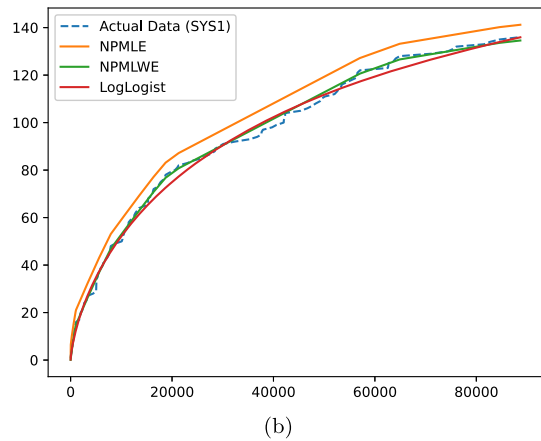
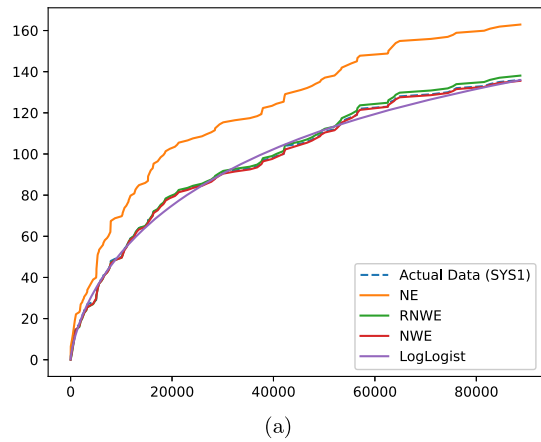


Table 5 Comparison of parametric and non-parametric methods in real data analyses

Data set	MAE				
Model name	DS1	DS2	DS3	DS4	DS5
<i>Parametric approach</i>					
Exp	7.31	1.71	3.36	6.61	NULL
Gamma	2.26	1.33	1.56	3.42	NULL
Pareto	2.38	1.04	1.18	3.12	NULL
TruncNormal	8.14	2.03	3.60	7.06	NULL
LogNormal	2.37	1.03	1.18	2.72	NULL
TruncLogist	7.36	1.82	3.40	6.67	NULL
LogLogist	1.95	1.10	1.31	3.03	NULL
TruncEVMMax	7.43	1.82	3.41	6.65	NULL
LogEVMMax	2.69	1.03	1.09	2.58	NULL
TruncEVMMin	8.29	2.14	3.60	7.15	NULL
LogEVMMin	2.04	1.26	1.41	3.14	NULL
Power law	NULL	NULL	NULL	NULL	5.24
Cow-lewis	NULL	NULL	NULL	NULL	4.10
<i>Existing non-parametric approach</i>					
NE	18.83	2.52	0.25	16.51	14.71
NPMLE	6.79	1.18	1.13	1.80	17.98
KBE (Gaussian)	4.06	7.04	3.96	6.24	0.84
KBE (Epanechnikov)	10.51	6.75	3.45	7.08	1.99
KBE (Tricube)	7.16	6.87	3.51	6.84	1.93
KBE (Triangle)	9.74	6.70	3.94	6.41	1.77
KBE (Quartic)	6.77	6.78	3.81	6.66	1.80
Best RNWE	0.96	0.26	0.34	0.42	0.35
<i>Our wavelet approach</i>					
Best NWE	0.67	0.11	0.37	0.39	0.26
Best NPMLWE	1.16	0.72	0.95	1.01	10.94
Best KBWE (Gaussian)	4.31	7.18	4.04	6.41	0.92
Best KBWE (Epanechnikov)	10.73	6.89	3.54	7.25	2.04
Best KBWE (Tricube)	7.39	7.00	3.61	7.01	1.98
Best KBWE (Triangle)	9.96	6.84	4.03	6.59	1.83
Best KBWE (Quartic)	7.00	6.92	3.90	6.83	1.86

Bold value indicates the best result in Parametric approach, Existing non-parametric approach or Our wavelet approach

parameters m and r carefully, comparing with thirteen parametric models and the existing non-parametric methods in many cases.

Finally, we investigate the sensitivity of model parameters m and r in the wavelet estimation approaches in the real data analyses. Figure 5 depicts the dependence of the model parameters m and r in the cumulative number of failures/repairs in DS1, when NPMLWE is applied. In Fig. 5a and b, NPMLWE ($r = 0, m = 2$) and NPMLWE ($r = 0.7, m = 10$) show the different behaviors from the other estimates, respectively. Similar to the simulation experiments, we confirm that MAE in the NPMLWE decreases gradually as the parameter r

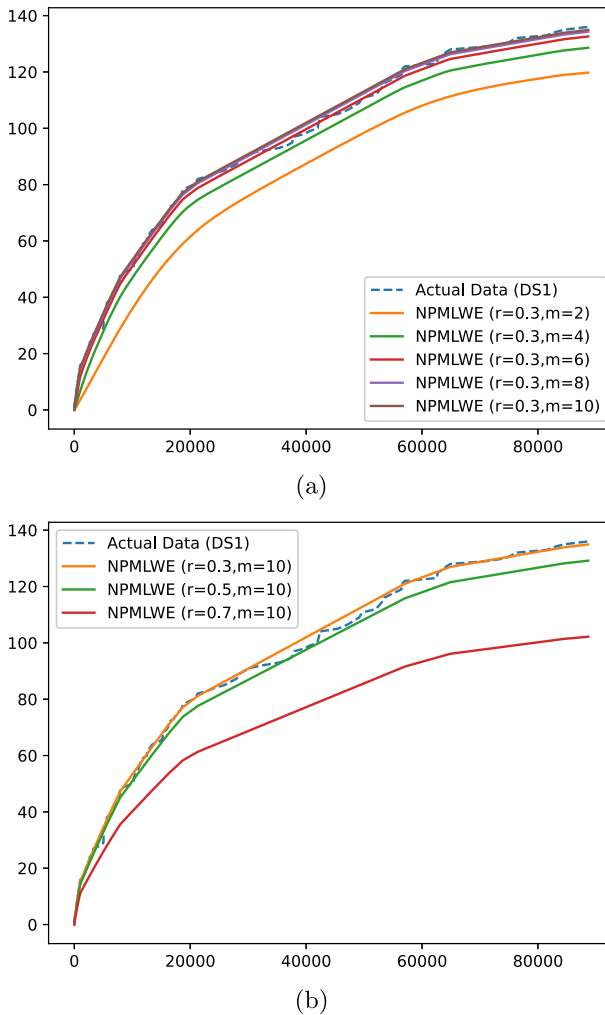


Fig. 5 Dependence of the model parameters m and r in the cumulative number of failures/repairs in DS1

decreases and the parameter m increases. To examine the dependence of the model parameters more precisely, we present the estimation results based on MAE with varying m and r in Tables 6, 7, 8, 9, and 10. Carefully checking the monotone properties with respect to m and r , it can be found that the mean absolute error MAE gradually decreases as the parameter r decreases. However, it should be noted that there are a few cases where MAE with $r = 0.5$ is smaller than that with $r = 0.3$ with the exactly same resolution level m . Hence, we recognize that the sensitivity of the resolution level m in the real data analyses is consistent with that in the simulation experiments. Through our data analyses, we can empirically know that the range of resolution level m should be tuned as $m \geq 7$.

Table 6 Sensitivity of model parameters m and r in the wavelet estimation approaches in DS1

$m \backslash r$	0.3	0.5	0.7	$m \backslash r$	0.3	0.5	0.7
<i>RNWE</i>				<i>NWE</i>			
2	14.88	21.83	33.69	2	14.78	22.47	36.10
3	9.90	14.83	26.29	3	9.83	15.81	29.62
4	6.76	10.22	20.90	4	6.70	11.47	25.09
5	4.53	7.30	17.54	5	4.48	8.68	22.24
6	2.85	5.17	15.34	6	2.82	6.67	20.37
7	1.56	3.54	13.62	7	1.71	5.18	18.96
8	1.45	2.21	12.37	8	0.67	4.22	17.99
9	8.79	0.96	11.47	9	1.11	3.40	17.44
10	34.57	9.27	9.23	10	5.02	1.49	16.69
<i>NPMLWE</i>				<i>KBWE (Gaussian)</i>			
2	13.87	21.67	35.50	2	15.09	22.68	36.24
3	8.71	14.89	28.95	3	10.29	16.10	29.82
4	5.39	10.34	24.29	4	7.39	11.88	25.34
5	3.14	7.42	21.30	5	5.69	9.38	22.65
6	1.88	5.35	19.34	6	4.80	7.93	21.05
7	1.32	3.92	17.88	7	4.45	7.25	20.19
8	1.18	3.11	16.93	8	4.34	7.01	19.86
9	1.16	2.69	16.40	9	4.32	6.95	19.77
10	1.23	2.42	16.11	10	4.31	6.94	19.75
<i>KBWE (Epanechnikov)</i>				<i>KBWE (tricube)</i>			
2	16.59	23.32	36.47	2	15.49	22.83	36.30
3	12.90	17.53	30.45	3	11.01	16.43	29.96
4	11.37	14.56	26.77	4	8.69	12.57	25.66
5	10.90	13.49	25.22	5	7.77	10.74	23.34
6	10.77	13.19	24.77	6	7.49	10.10	22.41
7	10.74	13.12	24.66	7	7.41	9.94	22.15
8	10.74	13.10	24.63	8	7.40	9.90	22.10
9	10.73	13.10	24.63	9	7.39	9.89	22.09
10	10.73	13.10	24.63	10	7.39	9.89	22.09
<i>KBWE (Triangle)</i>				<i>KBWE (Quartic)</i>			
2	16.34	23.21	36.43	2	15.42	22.81	36.29
3	12.47	17.28	30.34	3	10.89	16.37	29.93
4	10.77	14.11	26.51	4	8.49	12.46	25.60
5	10.19	12.87	24.80	5	7.46	10.53	23.22
6	10.02	12.49	24.24	6	7.12	9.79	22.20
7	9.97	12.39	24.08	7	7.03	9.58	21.88
8	9.96	12.36	24.05	8	7.01	9.53	21.81
9	9.96	12.36	24.04	9	7.00	9.52	21.79
10	9.96	12.36	24.04	10	7.00	9.52	21.79

Bold value indicates the best result in RNWE/NWE/NPMLWE/KBWE(Gaussian)/KBWE(Epanechnikov)/KBWE (Tricube)/KBWE(Triangle)/KBWE(Quartic)

Table 7 Sensitivity of model parameters m and r in the wavelet estimation approaches in DS2

m\r	0.3	0.5	0.7	m\r	0.3	0.5	0.7
<i>RNWE</i>				<i>NWE</i>			
2	4.72	7.92	15.65	2	4.58	8.52	17.87
3	2.77	4.86	11.86	3	2.64	5.66	14.69
4	1.64	3.11	9.74	4	1.51	3.99	12.90
5	1.01	2.10	8.44	5	0.88	3.06	11.85
6	0.62	1.53	7.77	6	0.52	2.52	11.28
7	0.26	1.15	7.33	7	0.33	2.19	10.92
8	1.78	0.61	7.07	8	0.19	2.03	10.73
9	8.77	1.58	6.66	9	0.11	1.92	10.64
10	27.70	9.14	4.61	10	0.19	1.83	10.54
<i>NPMLWE</i>				<i>KBWE (Gaussian)</i>			
2	3.97	8.10	17.60	2	8.21	10.97	19.18
3	1.86	5.03	14.30	3	7.47	9.37	17.02
4	0.86	3.21	12.35	4	7.26	8.86	16.23
5	0.72	2.19	11.17	5	7.20	8.72	16.00
6	0.80	1.72	10.49	6	7.19	8.68	15.94
7	0.90	1.52	10.12	7	7.18	8.67	15.93
8	0.96	1.43	9.92	8	7.18	8.67	15.92
9	0.99	1.39	9.83	9	7.18	8.67	15.92
10	0.99	1.38	9.78	10	7.18	8.67	15.92
<i>KBWE (Epanechnikov)</i>				<i>KBWE (tricube)</i>			
2	7.92	10.71	19.02	2	8.00	10.77	19.05
3	7.17	9.09	16.79	3	7.27	9.17	16.85
4	6.96	8.58	16.01	4	7.07	8.68	16.08
5	6.91	8.44	15.78	5	7.02	8.55	15.86
6	6.89	8.40	15.72	6	7.01	8.51	15.81
7	6.89	8.39	15.70	7	7.01	8.50	15.79
8	6.89	8.39	15.70	8	7.00	8.50	15.79
9	6.89	8.39	15.70	9	7.00	8.50	15.79
10	6.89	8.39	15.70	10	7.00	8.50	15.79
<i>KBWE (Triangle)</i>				<i>KBWE (Quartic)</i>			
2	7.93	10.74	19.04	2	7.95	10.75	19.04
3	7.15	9.09	16.81	3	7.20	9.12	16.82
4	6.92	8.55	16.00	4	6.99	8.61	16.03
5	6.86	8.40	15.75	5	6.94	8.47	15.80
6	6.85	8.36	15.69	6	6.92	8.43	15.74
7	6.84	8.35	15.67	7	6.92	8.42	15.73
8	6.84	8.34	15.66	8	6.92	8.42	15.72
9	6.84	8.34	15.66	9	6.92	8.42	15.72
10	6.84	8.34	15.66	10	6.92	8.42	15.72

Bold value indicates the best result in RNWE/NWE/NPMLWE/KBWE(Gaussian)/KBWE(Epanechnikov)/KBWE (Tricube)/KBWE(Triangle)/KBWE(Quartic)

Table 8 Sensitivity of model parameters m and r in the wavelet estimation approaches in DS3

$m \backslash r$	0.3	0.5	0.7	$m \backslash r$	0.3	0.5	0.7
<i>RNWE</i>				<i>NWE</i>			
2	5.62	8.23	12.80	2	5.64	8.59	13.95
3	3.61	5.60	10.21	3	3.62	6.05	11.63
4	2.31	3.72	8.09	4	2.32	4.27	9.84
5	1.48	2.50	6.67	5	1.50	3.14	8.67
6	0.94	1.74	5.79	6	0.98	2.42	7.94
7	0.34	1.21	5.22	7	0.65	1.98	7.47
8	1.32	0.64	4.84	8	0.48	1.72	7.19
9	5.85	1.10	4.39	9	0.38	1.55	7.02
10	18.37	6.33	2.73	10	0.37	1.42	6.89
<i>NPMLWE</i>				<i>KBWE (Gaussian)</i>			
2	4.61	7.68	13.30	2	6.34	8.88	14.04
3	2.77	5.05	10.88	3	4.93	6.77	11.95
4	1.74	3.25	9.01	4	4.32	5.62	10.57
5	1.21	2.26	7.76	5	4.12	5.18	9.95
6	0.98	1.68	6.98	6	4.06	5.05	9.75
7	0.95	1.36	6.51	7	4.05	5.01	9.70
8	0.97	1.20	6.23	8	4.05	5.01	9.69
9	0.98	1.13	6.08	9	4.04	5.00	9.69
10	0.98	1.11	6.01	10	4.04	5.00	9.69
<i>KBWE (Epanechnikov)</i>				<i>KBWE (tricube)</i>			
2	6.11	8.77	14.00	2	6.14	8.78	14.00
3	4.56	6.52	11.83	3	4.60	6.55	11.84
4	3.85	5.23	10.33	4	3.91	5.27	10.35
5	3.62	4.71	9.61	5	3.68	4.77	9.64
6	3.56	4.56	9.37	6	3.62	4.63	9.42
7	3.54	4.53	9.32	7	3.61	4.59	9.37
8	3.54	4.52	9.30	8	3.61	4.59	9.36
9	3.54	4.52	9.30	9	3.61	4.58	9.35
10	3.54	4.52	9.30	10	3.61	4.58	9.35
<i>KBWE (Triangle)</i>				<i>KBWE (Quartic)</i>			
2	6.32	8.86	14.03	2	6.26	8.83	14.02
3	4.91	6.74	11.93	3	4.80	6.67	11.90
4	4.30	5.60	10.55	4	4.17	5.49	10.48
5	4.11	5.16	9.94	5	3.98	5.04	9.84
6	4.05	5.04	9.74	6	3.92	4.91	9.64
7	4.03	5.00	9.69	7	3.91	4.88	9.59
8	4.03	4.99	9.68	8	3.91	4.87	9.58
9	4.03	4.99	9.68	9	3.90	4.87	9.58
10	4.03	4.99	9.68	10	3.90	4.87	9.58

Bold value indicates the best result in RNWE/NWE/NPMLWE/KBWE(Gaussian)/KBWE(Epanechnikov)/KBWE (Tricube)/KBWE(Triangle)/KBWE(Quartic)

Table 9 Sensitivity of model parameters m and r in the wavelet estimation approaches in DS4

$m \backslash r$	0.3	0.5	0.7	$m \backslash r$	0.3	0.5	0.7
<i>RNWE</i>				<i>NWE</i>			
2	12.49	18.35	27.39	2	12.37	18.78	29.06
3	7.52	12.27	21.54	3	7.38	12.83	23.81
4	4.35	7.50	16.46	4	4.23	8.34	19.42
5	2.70	4.67	12.81	5	2.58	5.77	16.53
6	1.74	3.22	10.90	6	1.64	4.40	14.95
7	0.98	2.33	9.92	7	0.94	3.56	14.12
8	0.42	1.58	9.22	8	0.39	2.91	13.54
9	2.95	0.51	8.62	9	0.64	2.31	13.08
10	13.80	2.93	7.67	10	2.32	1.45	12.60
<i>NPMLWE</i>				<i>KBWE (Gaussian)</i>			
2	11.60	18.16	28.64	2	13.48	19.30	29.30
3	6.52	12.14	23.31	3	9.50	14.02	24.36
4	3.20	7.50	18.83	4	7.43	10.54	20.61
5	1.59	4.70	15.73	5	6.70	8.97	18.53
6	1.03	3.15	14.02	6	6.48	8.48	17.78
7	1.01	2.32	13.05	7	6.43	8.34	17.57
8	1.17	1.86	12.43	8	6.42	8.31	17.51
9	1.32	1.64	12.03	9	6.41	8.30	17.50
10	1.40	1.55	11.79	10	6.41	8.30	17.50
<i>KBWE (Epanechnikov)</i>				<i>KBWE (tricube)</i>			
2	13.70	19.38	29.33	2	13.62	19.35	29.32
3	9.93	14.24	24.44	3	9.78	14.16	24.41
4	8.09	11.00	20.84	4	7.88	10.84	20.76
5	7.48	9.64	18.97	5	7.24	9.43	18.82
6	7.31	9.24	18.35	6	7.07	9.01	18.18
7	7.26	9.13	18.19	7	7.03	8.91	18.01
8	7.25	9.11	18.14	8	7.02	8.88	17.97
9	7.25	9.10	18.14	9	7.01	8.88	17.96
10	7.25	9.10	18.13	10	7.01	8.87	17.96
<i>KBWE (Triangle)</i>				<i>KBWE (Quartic)</i>			
2	13.52	19.31	29.31	2	13.58	19.33	29.31
3	9.58	14.05	24.37	3	9.70	14.12	24.39
4	7.58	10.63	20.65	4	7.75	10.75	20.71
5	6.87	9.12	18.62	5	7.08	9.30	18.74
6	6.66	8.64	17.91	6	6.89	8.85	18.06
7	6.61	8.51	17.70	7	6.84	8.74	17.87
8	6.59	8.48	17.65	8	6.83	8.71	17.83
9	6.59	8.47	17.64	9	6.83	8.70	17.82
10	6.59	8.47	17.63	10	6.83	8.70	17.82

Bold value indicates the best result in RNWE/NWE/NPMLWE/KBWE(Gaussian)/KBWE(Epanechnikov)/KBWE (Tricube)/KBWE(Triangle)/KBWE(Quartic)

Table 10 Sensitivity of model parameters m and r in the wavelet estimation approaches in DS5

$m \backslash r$	0.3	0.5	0.7	$m \backslash r$	0.3	0.5	0.7
<i>RNWE</i>				<i>NWE</i>			
2	2.13	3.99	7.90	2	2.13	4.27	10.35
3	1.14	2.09	6.28	3	1.08	2.61	9.67
4	0.78	1.37	6.03	4	0.67	2.05	9.27
5	0.62	1.10	5.97	5	0.45	1.79	9.05
6	0.55	1.03	5.98	6	0.33	1.70	8.93
7	0.35	0.96	5.96	7	0.26	1.64	8.85
8	1.70	0.55	5.90	8	0.28	1.55	8.79
9	7.35	1.30	5.57	9	1.04	1.26	8.71
10	24.43	7.95	3.74	10	3.07	0.42	8.41
<i>NPMLWE</i>				<i>KBWE (Gaussian)</i>			
2	16.26	13.92	10.94	2	2.27	4.36	10.52
3	17.01	14.80	10.97	3	1.36	2.85	9.89
4	17.39	15.27	11.06	4	1.08	2.40	9.51
5	17.59	15.52	11.14	5	0.97	2.22	9.32
6	17.68	15.66	11.18	6	0.93	2.16	9.24
7	17.73	15.73	11.21	7	0.92	2.13	9.19
8	17.75	15.76	11.22	8	0.92	2.12	9.17
9	17.76	15.78	11.23	9	0.92	2.11	9.15
10	17.76	15.78	11.23	10	0.92	2.11	9.15
<i>KBWE (Epanechnikov)</i>				<i>KBWE (tricube)</i>			
2	3.07	4.83	10.58	2	3.02	4.80	10.58
3	2.35	3.48	9.94	3	2.28	3.43	9.93
4	2.12	3.01	9.60	4	2.06	2.96	9.59
5	2.07	2.86	9.43	5	2.00	2.82	9.42
6	2.05	2.82	9.35	6	1.98	2.78	9.34
7	2.05	2.81	9.31	7	1.98	2.77	9.30
8	2.04	2.80	9.29	8	1.98	2.76	9.28
9	2.04	2.80	9.28	9	1.98	2.76	9.27
10	2.04	2.80	9.28	10	1.98	2.76	9.27
<i>KBWE (Triangle)</i>				<i>KBWE (Quartic)</i>			
2	2.93	4.74	10.57	2	2.94	4.75	10.57
3	2.16	3.35	9.93	3	2.18	3.36	9.93
4	1.92	2.87	9.58	4	1.94	2.88	9.58
5	1.85	2.73	9.41	5	1.88	2.75	9.41
6	1.83	2.68	9.33	6	1.86	2.70	9.33
7	1.83	2.66	9.29	7	1.86	2.69	9.29
8	1.83	2.66	9.27	8	1.86	2.68	9.27
9	1.83	2.65	9.26	9	1.86	2.68	9.26
10	1.83	2.65	9.25	10	1.86	2.68	9.25

Bold value indicates the best result in RNWE/NWE/NPMLWE/KBWE(Gaussian)/KBWE(Epanechnikov)/KBWE(Tricube)/KBWE(Triangle)/KBWE(Quartic)

7 Conclusions

In this paper, we have developed novel non-parametric estimation methods based on the Daubechies wavelets for failure/repair time data arising in the lifetime analysis. More specifically, we have revisited Kuhl and Bhairgond estimate (Kuhl & Bhairgond, 2000), and given the exact solution based on the naive estimate of the intensity function for NHPP and alternative methods based on the non-parametric maximum likelihood method by Boswell (1966) and kernel-based estimates. Throughout simulation experiments, we have compared our Daubechies wavelet approaches with the classical maximum likelihood estimation for two parametric models and the existing four non-parametric estimation methods. The results of the simulation experiments showed that when we completely know the real mean value function, the power-law model achieved the smallest mean absolute error, which is also the theoretically optimal result. Our proposed NPMLWE method provided the second best result even though the real model (intensity function) to describe the failure/repair process is not known. We also tested the goodness-of-fit performance of 13 parametric models, 4 existing non-parametric models, and 3 wavelet methods with five data sets. The experiments with real data suggested that our proposed wavelet method can achieve the best goodness-of-fit performance in most cases. We also examined the sensitivity of turning parameter m and resolution parameter r in the wavelet-based approach, and showed that in most cases, as r decreases, the mean absolute error, which is a criterion on the goodness-of-fit, decreases. As a practical suggestion in using the wavelet-based approach in lifetime analysis of repairable systems for m , we could empirically suggest to set the range of resolution level as $m \geq 7$.

In the future, we will develop a prediction method by means of the wavelet approach. Since the Daubechies wavelet-based inference is a non-parametric method, it is difficult to assess the predictive lifetime distribution. Recently, Wu et al. (2021) developed W-SRAT: Wavelet-based software reliability assessment tool, which is based on the discrete Haar wavelet, and implemented an idea to continue sequential predictions in the wavelet shrinkage estimation. The similar idea may be applied to the lifetime analysis with the Daubechies wavelets.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest. This article does not contain any studies involving human participants performed by any of the authors.

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